Conformal mapping for cavity inverse problem: an explicit reconstruction formula

Alexandre Munnier, Karim Ramdani

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Abstract

In this paper, we address a classical case of the Calderón (or conductivity) inverse problem in dimension two. We aim to recover the location and the shape of a single cavity ω (with boundary γ) contained in a domain Ω (with boundary Γ) from the knowledge of the Dirichlet-to-Neumann (DtN) map $\Lambda_\gamma : f \longmapsto \partial_n u^f|_\Gamma$, where u^f is harmonic in $\Omega \setminus \overline{\omega}$, $u^f|_\Gamma = f$ and $u^f|_\gamma = c^f$, c^f being the constant such that $\int_\gamma \partial_n u^f \,\mathrm{d}s = 0$. We obtain an explicit formula for the complex coefficients a_m arising in the expression of the Riemann map $z \longmapsto a_1 z + a_0 + \sum_{m\leqslant -1} a_m z^m$ that conformally maps the exterior of the unit disk onto the exterior of ω. This formula is derived by using two ingredients: a new factorization result of the DtN map and the so-called generalized Pólia-Szegö tensors (GPST) of the cavity. As a byproduct of our analysis, we also prove the analytic dependence of the coefficients a_m with respect to the DtN. Numerical results are provided to illustrate the efficiency and simplicity of the method.

1 Introduction

Let Ω be a simply connected open bounded set in $\mathbb{R}^2$ with Lipschitz boundary Γ. Let σ be a positive function in $L^\infty(\Omega)$ and consider the elliptic boundary value problem:

$$-\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \tag{1.1a}$$
$$u = f \quad \text{on } \Gamma. \tag{1.1b}$$

Calderón's inverse conductivity problem [16] can be stated as follows: Knowing the Dirichlet-to-Neumann (DtN) map $\Lambda : f \longmapsto \partial_n u^f$, is it possible to recover the conductivity σ?

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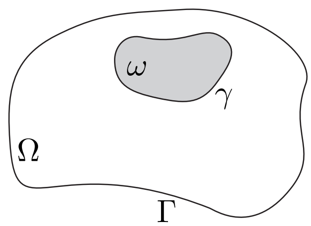

Figure 1: The geometry.

In this work, we focus on the particular case of piecewise conductivity with infinitely high contrast (see for instance Friedman and Vogelius [20] who considered this problem in the case of small inclusions). More precisely, we suppose that Ω contains a cavity ω, where ω is an open simply connected domain with Lipschitz boundary γ and such that $\overline{\omega} \subset \Omega$ (see Figure 1). We denote by n the unit normal to $\Gamma \cup \gamma$ directed towards the exterior of $\Omega \setminus \overline{\omega}$.

For every f in $H^{\frac{1}{2}}(\Gamma)$, we denote by $(u^f, c^f) \in H^1(\Omega \setminus \overline{\omega}) \times \mathbb{R}$ the solution to the Dirichlet problem:

$$-\Delta u^f = 0 \quad \text{in } \Omega \setminus \overline{\omega} \tag{1.2a}$$

$$u^f = f \quad \text{on } \Gamma \tag{1.2b}$$

$$u^f = c^f \quad \text{on } \gamma, \tag{1.2c}$$

where c^f is the unique constant such that:

$$\int_\gamma \partial_n u^f \, \mathrm{d}\sigma = 0. \tag{1.2d}$$

Problem (1.2) is well-posed and its solution is the limit of the solution of (1.1) for piecewise constant conductivity, when the contrast between the cavity and the background tends to infinity (see Proposition A.1 of the Appendix for a precise statement of this classical result and for the proof, which is given for the sake of completeness).

Loosely speaking (the exact functional framework will be made precise later on), the inverse problem considered throughout this paper is the following: *knowing the Dirichlet-to-Neumann (DtN) map $\Lambda_\gamma : f \longmapsto \partial_n u^f$, how to reconstruct the cavity ω?*

Remark 1.1 *In dimension 2, it is classical to see u^f as the harmonic conjugate function of v^g, the solution to (defined up to a constant):*

$$-\Delta v^g = 0 \quad \textit{in } \Omega \setminus \overline{\omega} \tag{1.3a}$$

$$\partial_n v^g = g \quad \textit{on } \Gamma \tag{1.3b}$$

$$\partial_n v^g = 0 \quad \textit{on } \gamma, \tag{1.3c}$$

where $g = \partial_\tau f$ and $\tau := n^\perp$ is the unit tangent vector to Γ. The function u^f is usually referred to as the stream function associated with the potential function v^g and we have $\partial_n u^f = -\partial_\tau v^g$ on Γ. According to the above relations, the knowledge of Λ_γ (i.e. the DtN for u^f) is equivalent to the knowledge of the Neumann-to-Dirichlet map $\tilde{\Lambda}_\gamma : g \longmapsto v^g$ for problem (1.3)*, as we have $\Lambda_\gamma f = -\partial_\tau \tilde{\Lambda}_\gamma(\partial_\tau f)$.*

Classically for inverse problems, the questions of uniqueness, stability and reconstruction have been studied in the literature for cavities identification. Regarding uniqueness, it is well-known that one pair $(f, \partial_n u^f)$ of Cauchy data uniquely determines the geometry of the cavity for a Dirichlet boundary condition (see Kress [36]) or a Neumann boundary condition (see Alessandrini and Rondi [2]). For Robin type condition, Bacchelli [10] proved that two excitations f_1 and f_2 uniquely determine the cavity provided they are linearly independent and one of them is positive. Concerning stability, as shown by Mandache [39], logarithmic stability is best possible (see also Alessandrini and Rondi [2] and references therein). Among the reconstruction methods available in the literature for shape identification, one can distinguish two classes of approaches: iterative and non iterative methods (see for instance the survey paper by Potthast [42] for an overview of reconstruction methods). In the first class of methods, one computes a sequence of approximating shapes, generally by solving at each step the direct problem and using minimal data (typically only one or several pairs of Cauchy data, and not the full DtN map). Among these approaches, we can mention those based on optimization [11, 17], on the reciprocity gap principle [38, 33, 15], on the quasi-reversibility [12, 13] or on conformal mapping [1, 36, 22, 23, 24, 37, 25].

The second class of methods covers non iterative methods which are generally based on the construction (from the measurements) of an indicator function of the inclusion(s). These sampling/probe methods do not need to solve the forward problem, but require the knowledge of the full DtN map. Among these reconstruction techniques, let us mention –with no claim as to completeness– the enclosure and probe method of Ikehata [29, 31, 30, 32, 19], Kirsch's Factorization method [14, 26, 35] and Generalized Polya-Szegö Tensors [6, 7, 8, 5, 34].

Our purpose in this paper is to propose a new non iterative reconstruction method that combines some of the ingredients used in earlier works, namely: a new factorization result (Theorem 3.1), Generalized Polya-Szegö Tensors and conformal mapping. The main feature of our reconstruction method is that we end up with an *explicit reconstruction formula* (Theorem 3.4) for the complex coefficients a_k arising in the expression of the Riemann map $z \longmapsto a_1 z + a_0 + \sum_{m\leqslant -1} a_m z^m$ that conformally maps the exterior of the unit disk onto the exterior of ω. Let us emphasize that these reconstruction formulae also yield the analytic dependence of the coefficients with respect to the DtN.

The closest results to our article are those obtained by Kang *et al.* [34]. Although being close to the second part of our contribution (the recovery of the conformal map from the GPST), our improvements are twofold. First, we do not assume the GPST to be given. Indeed, the main advantage of our new factorization is that it allows one to recover very accurately these GPST from the DtN measurements. Second, even assuming the GPST to be known, Kang *et al.* obtain a recursive formula for the coefficients of the conformal mapping in terms of the coefficients of the GPST. In our work, we obtain for each of these coefficients an explicit and non recursive formula.

The proposed reconstruction algorithm can –in principle– be adapted to other boundary conditions. However, such as most direct reconstruction methods, it requires the knowledge of the full DtN map and so far, it is limited to the two-dimensional case due to the use of conformal mapping.

The paper is organized as follows: we present in Section 2 a boundary integral formulation of the problem. Section 3 is devoted to the derivation of the reconstruction formula, using a new factorization result and GPST. Some issues about stability are also discusses therein. Finally, some numerical results are given in Section 4.

2 Boundary integral formulation

2.1 Background on single layer potential

In this section, we collect some well known facts of potential theory, and more especially on single layer potential, that are crucial for our method. For more details and for the proofs, we refer the interested reader to the monographs of McLean [40], Steinbach [43] or Hsiao and Wendland [28].

Throughout the article, we shall denote by

$$G(x) = -\frac{1}{2\pi}\log|x|$$

the fundamental solution of the operator $-\Delta$ in $\mathbb{R}^2$.

Let C_i be a bounded, Lipschitz domain (see [40, Definition 3.28]) and denote by $\mathscr{C}$ its boundary. Let n be the unit normal to $\mathscr{C}$ directed towards the exterior of C_i.

The exterior of C_i is denoted $\mathsf{C}_e := \mathbb{R}^2 \setminus \overline{\mathsf{C}}_i$. Given a function u defined on $\mathbb{R}^2 \setminus \mathscr{C}$, we denote by u_i and u_e its restrictions respectively to C_i and C_e. Whenever the traces and the normal traces of u_i and u_e on $\mathscr{C}$ exist respectively in $H^{\frac{1}{2}}(\mathscr{C})$ and $H^{-\frac{1}{2}}(\mathscr{C})$, we denote by $[u]_{\mathscr{C}} = u_i|_{\mathscr{C}} - u_e|_{\mathscr{C}} \in H^{\frac{1}{2}}(\mathscr{C})$ the jump of u across $\mathscr{C}$ and by $[\partial_n u]_{\mathscr{C}} = (\partial_n u_i)|_{\mathscr{C}} - (\partial_n u_e)|_{\mathscr{C}} \in H^{-\frac{1}{2}}(\mathscr{C})$ the jump of the normal traces of u across $\mathscr{C}$.

Definition 2.1 *For every $\hat{q} \in H^{-\frac{1}{2}}(\mathscr{C})$, we denote by $\mathscr{S}_{\mathscr{C}}\hat{q}$ the single layer potential associated with the density $\hat{q}$.*

The single layer potential $\mathscr{S}_{\mathscr{C}}\hat{q}$ defines a harmonic function in $\mathbb{R}^2 \setminus \mathscr{C}$. The operator $\mathscr{S}_{\mathscr{C}}$ is an integral operator with weakly singular kernel, so that for $\hat{q} \in L^\infty(\mathscr{C})$ for instance and $x \in \mathbb{R}^2 \setminus \mathscr{C}$, it reads:

$$\mathscr{S}_{\mathscr{C}}\hat{q}(x) = \int_{\mathscr{C}} G(x-y)\hat{q}(y)\,\mathrm{d}\sigma_y.$$

Moreover, the single layer potential defines a bounded linear operator from $H^{-\frac{1}{2}}(\mathscr{C})$ into $H^1_{\ell\mathrm{oc}}(\mathbb{R}^2)$, and $\mathscr{S}_{\mathscr{C}}\hat{q}$ admits the following asymptotic behavior at infinity (see for instance [40, p. 261])

$$\mathscr{S}_{\mathscr{C}}\hat{q}(x) = -\frac{1}{2\pi}\langle \hat{q}, 1\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} \log|x| + O(|x|^{-1}), \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}}$ stands for the duality brackets between $H^{-\frac{1}{2}}(\mathscr{C})$ and $H^{\frac{1}{2}}(\mathscr{C})$. This shows in particular that $\nabla(\mathscr{S}_{\mathscr{C}}\hat{q})$ belongs to L^2 if and only if $\hat{q}$ belongs to the function space

$$\widehat{H}(\mathscr{C}) := \{\hat{q} \in H^{-\frac{1}{2}}(\mathscr{C}) \,:\, \langle \hat{q}, 1\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} = 0\}.$$

We also recall that the single layer potential satisfies the following classical jump conditions

$$[\mathscr{S}_{\mathscr{C}}\hat{q}\,]_{\mathscr{C}} = 0, \qquad\qquad [\partial_n(\mathscr{S}_{\mathscr{C}}\hat{q})]_{\mathscr{C}} = \hat{q}. \tag{2.2}$$

Let us focus now on the trace of the single layer potential.

Definition 2.2 *For every $\hat{q} \in H^{-\frac{1}{2}}(\mathscr{C})$, we denote by $\mathsf{S}_{\mathscr{C}}\hat{q}$ the trace of the single layer operator $\mathscr{S}_{\mathscr{C}}\hat{q}$ on $\mathscr{C}$.*

The operator $\mathsf{S}_{\mathscr{C}}$ is an integral operator with weakly singular kernel as well. For $\hat{q} \in L^\infty(\mathscr{C})$ and for every $x \in \mathscr{C}$, it reads:

$$\mathsf{S}_{\mathscr{C}}\hat{q}(x) = \int_{\mathscr{C}} G(x-y)\hat{q}(y)\,\mathrm{d}\sigma_y.$$

The trace $\mathsf{S}_{\mathscr{C}}$ of the single layer operator defines a bounded linear operator from $H^{-\frac{1}{2}}(\mathscr{C})$ into $H^{\frac{1}{2}}(\mathscr{C})$. Furthermore, using Green's formula and the asymptotics (2.1), we can easily prove the identity

$$\langle \hat{q}, \mathsf{S}_\Gamma\hat{q}\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} = \int_{\mathbb{R}^2} |\nabla(\mathscr{S}_{\mathscr{C}}\hat{q})|^2\,\mathrm{d}x < +\infty, \qquad \forall \hat{q} \in \widehat{H}(\mathscr{C}). \tag{2.3}$$

Following [40, Theorem 8.15], we also introduce the following particular density and constant which will play a crucial role in our analysis.

Definition 2.3 *The equilibrium density for $\mathscr{C}$ is the unique density $\hat{\mathsf{e}}_{\mathscr{C}} \in H^{-\frac{1}{2}}(\mathscr{C})$ such that $\mathsf{S}_{\mathscr{C}}\hat{\mathsf{e}}_{\mathscr{C}}$ is constant on $\mathscr{C}$ and*

$$\langle \hat{\mathsf{e}}_{\mathscr{C}}, 1\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} = 1.$$

The logarithmic capacity $\mathrm{Cap}(\mathscr{C})$ of $\mathscr{C}$ is defined as being the positive constant:

$$\mathrm{Cap}(\mathscr{C}) = \exp\left(-2\pi\mathsf{S}_{\mathscr{C}}\hat{\mathsf{e}}_{\mathscr{C}}\right).$$

Setting

$$H(\mathscr{C}) := \{q \in H^{\frac{1}{2}}(\mathscr{C}) \,:\, \langle \hat{\mathsf{e}}_{\mathscr{C}}, q\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} = 0\}$$

we know, following McLean [40], that the linear operator:

$$\mathsf{S}_{\mathscr{C}} : \hat{q} \in \widehat{H}(\mathscr{C}) \longmapsto q \in H(\mathscr{C}),$$

defines an isomorphism that extends into an isomorphism from $H^{-\frac{1}{2}}(\mathscr{C})$ onto $H^{\frac{1}{2}}(\mathscr{C})$, if $\mathrm{Cap}(\mathscr{C}) \neq 1$ (see [40, Theorem 8.16]). Under this condition, one can identify via this isomorphism any density $\hat{q} \in \widehat{H}(\mathscr{C})$ with the trace

$$q := \mathsf{S}_{\mathscr{C}}\hat{q} \in H(\mathscr{C}).$$

This identification will be systematically used throughout the paper, using the notation with (respectively without) a hat on the density like quantities (respectively on traces). This isomorphism turns out to be an isometry provided the spaces $\widehat{H}(\mathscr{C})$ and $H(\mathscr{C})$ are endowed with the following inner products:

Definition 2.4 *For all $\hat{q}, \hat{p} \in \widehat{H}(\mathscr{C})$, we set:*

$$\langle \hat{q}, \hat{p}\rangle_{-\frac{1}{2},\mathscr{C}} = \langle q, p\rangle_{\frac{1}{2},\mathscr{C}} = \langle \hat{q}, p\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}}.$$

According to (2.3), the inner products introduced in Definition 2.4 are related to the Dirichlet energy of the single layer potential through the following identities:

$$\|\hat{q}\|^2_{-\frac{1}{2},\mathscr{C}} = \|q\|^2_{\frac{1}{2},\mathscr{C}} = \int_{\mathbb{R}^2} |\nabla(\mathscr{S}_{\mathscr{C}}\hat{q})|^2 \,\mathrm{d}x, \qquad \forall \hat{q} \in \widehat{H}(\mathscr{C}).$$

We also need in the sequel the following orthogonal projections.

Definition 2.5 *Let $\Pi_{\mathscr{C}}$ and $\widehat{\Pi}_{\mathscr{C}}$ denote respectively the orthogonal projections from $H^{\frac{1}{2}}(\mathscr{C})$ into $H(\mathscr{C})$ and from $H^{-\frac{1}{2}}(\mathscr{C})$ into $\widehat{H}(\mathscr{C})$.*

In particular, we have the following unique decompositions:

$$\begin{aligned}
\forall\, \hat{q} \in H^{-\frac{1}{2}}(\mathscr{C}) : &\qquad \hat{q} = \langle \hat{q}, 1\rangle\, \hat{\mathsf{e}}_{\mathscr{C}} + \hat{q}_0, &\qquad \hat{q}_0 := \widehat{\Pi}_{\mathscr{C}}\hat{q} \in \widehat{H}(\mathscr{C}),\\
\forall\, q \in H^{\frac{1}{2}}(\mathscr{C}) : &\qquad q = \langle \hat{\mathsf{e}}_{\mathscr{C}}, q\rangle\, 1 + q_0, &\qquad q_0 := \Pi_{\mathscr{C}} q \in H(\mathscr{C}).
\end{aligned}$$

Definition 2.6 *We denote by $\mathsf{Tr}_{\mathscr{C}}$ the classical trace operator (valued into $H^{\frac{1}{2}}(\mathscr{C})$), and by $\mathsf{Tr}^0_{\mathscr{C}}$ when it is left-composed with the orthogonal projection onto $H(\mathscr{C})$: $\mathsf{Tr}^0_{\mathscr{C}} := \Pi_{\mathscr{C}}\mathsf{Tr}_{\mathscr{C}}$.*

Let us conclude this preliminary section by a useful characterization of the chosen norm on $H(\mathscr{C})$. Classically (see for instance [9, 21]), we define the quotient weighted Sobolev space:

$$W_0^1(\mathbb{R}^2) = \{u \in \mathcal{D}'(\mathbb{R}^2) \,:\, \rho u \in L^2(\mathbb{R}^2),\, \nabla u \in (L^2(\mathbb{R}^2))^2\}/\mathbb{R},$$

where the weight is given by

$$\rho(x) := \left(\sqrt{1+|x|^2}\log(2+|x|^2)\right)^{-1}, \qquad x \in \mathbb{R}^2,$$

and where the quotient means that functions of $W_0^1(\mathbb{R}^2)$ are defined up to an additive constant. This space is a Hilbert space once equipped with the inner product:

$$\langle u, v\rangle_{W_0^1(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \nabla u \cdot \nabla v \,\mathrm{d}x.$$

In particular, according to (2.3), $\mathscr{S}_{\mathscr{C}}\hat{q} \in W_0^1(\mathbb{R}^2)$ if and only if $q \in H(\mathscr{C})$, and moreover

$$\|\hat{q}\|_{-\frac{1}{2},\mathscr{C}} = \|q\|_{\frac{1}{2},\mathscr{C}} = \|\mathscr{S}_{\mathscr{C}}\hat{q}\|_{W_0^1(\mathbb{R}^2)}, \qquad \forall q \in H(\mathscr{C}).$$

Lemma 2.7 *For every $q \in H(\mathscr{C})$, we have*

$$\|q\|_{\frac{1}{2},\mathscr{C}} = \inf\left\{\|u\|_{W_0^1(\mathbb{R}^2)} \,:\, u \in W_0^1(\mathbb{R}^2) \text{ and } \mathsf{T}r^0_{\mathscr{C}} u = q\right\}.$$

The infimum is a minimum which is uniquely achieved by $u = \mathscr{S}_{\mathscr{C}}\hat{q}$.

Proof : Given $q \in H(\mathscr{C})$, let us consider the orthogonal decomposition:

$$W_0^1(\mathbb{R}^2) = \langle \mathscr{S}_{\mathscr{C}}\hat{q}\rangle \oplus \langle \mathscr{S}_{\mathscr{C}}\hat{q}\rangle^{\perp}.$$

Let $u \in W_0^1(\mathbb{R}^2)$ be such that $\mathsf{T}r^0_{\mathscr{C}} u = q$. Writing u in the form $u = \lambda\mathscr{S}_{\mathscr{C}}\hat{q} + v$ with $\lambda \in \mathbb{R}$ and $v \in \langle \mathscr{S}_{\mathscr{C}}\hat{q}\rangle^{\perp}$, and taking the projected trace on $\mathscr{C}$ we get:

$$q = \lambda q + \mathsf{T}r^0_{\mathscr{C}} v \qquad \text{in } H(\mathscr{C}).$$

Forming now the duality product with $\hat{q}$ and taking into account that:

$$\langle \hat{q}, \mathsf{T}r^0_{\mathscr{C}} v\rangle_{-\frac{1}{2},\frac{1}{2},\mathscr{C}} = \langle \mathscr{S}_{\mathscr{C}}\hat{q}, v\rangle_{W_0^1(\mathbb{R}^2)} = 0,$$

we deduce that $\lambda = 1$. Since we have now

$$\|u\|^2_{W_0^1(\mathbb{R}^2)} = \|q\|^2_{\frac{1}{2},\mathscr{C}} + \|v\|^2_{W_0^1(\mathbb{R}^2)},$$

with v such that $\mathsf{T}r^0_{\mathscr{C}} v = 0$, the conclusion follows. $\square$

2.2 Boundary interaction and single layer potential

In this section, we are interested in quantifying the Dirichlet energy variation between $\mathscr{S}_\Gamma\hat{q}$ and $\mathscr{S}_\gamma\hat{p}$ where $p = \mathsf{T}r^0_\gamma\mathscr{S}_\Gamma\hat{q}$ (i.e. p is the trace of the single layer potential $\mathscr{S}_\Gamma\hat{q}$ on γ).

Definition 2.8 *We define the boundary interaction operators K_Γ^γ and K_γ^Γ between Γ and γ by:*

$$\mathsf{K}_\Gamma^\gamma : q \in H(\Gamma) \longmapsto \mathsf{Tr}^0_\gamma(\mathscr{S}_\Gamma\hat{q}) \in H(\gamma), \qquad \mathsf{K}_\gamma^\Gamma : p \in H(\gamma) \longmapsto \mathsf{Tr}^0_\Gamma(\mathscr{S}_\gamma\hat{p}) \in H(\Gamma),$$

where Tr^0_γ and Tr^0_Γ are given in Definition 2.6.

The next result shows that Tr^0_Γ can be replaced by Tr_Γ in the definition of K_γ^Γ:

Lemma 2.9 *If $p \in H(\gamma)$, then $q := \mathsf{T}r_\Gamma(\mathscr{S}_\gamma\hat{p})$ belongs to $H(\Gamma)$.*

Proof : Let p and q be given as in the statement of the lemma and let us define the function $w := (w_i, w_e)$ in $H^1_{\ell oc}(\mathbb{R}^2)$ by setting: $w_i = \mathscr{S}_\Gamma\hat{q}$ in Ω and $w_e = \mathscr{S}_\gamma\hat{p}$ in $\mathbb{R}^2 \setminus \overline{\Omega}$. According to (2.3), the function w has finite Dirichlet energy since $p \in H(\gamma)$. Thanks to (2.1), we see that $w_e(x) = O(|x|^{-1})$ at infinity, and this allows us to obtain the following classical integral representation formula for every $x \in \mathbb{R}^2 \setminus \overline{\Omega}$ (see for instance [43, p. 182] or [18, Lemma 3.5])

$$\begin{aligned} w_e(x) &= \langle \partial_n G(x-\cdot), w_e\rangle_{\frac12,\Gamma} - \langle \partial_n w_e, G(x-\cdot)\rangle_{-\frac12,\frac12,\Gamma} \\ 0 &= \langle \partial_n G(x-\cdot), w_i\rangle_{-\frac12,\frac12,\Gamma} - \langle \partial_n w_i, G(x-\cdot)\rangle_{-\frac12,\frac12,\Gamma}. \end{aligned}$$

Since $\mathsf{T}r_\Gamma w_i = \mathsf{T}r_\Gamma w_e = q$, we get by subtracting these identities that:

$$w_e(x) = \langle \hat{r}, G(x-\cdot)\rangle_{-\frac12,\frac12,\Gamma} = \mathscr{S}_\Gamma\hat{r}(x), \qquad x \in \mathbb{R}^2 \setminus \overline{\Omega},$$

where the density $\hat{r} := \partial_n w_i - \partial_n w_e$ belongs to $\widehat{H}(\Gamma)$ since, as already mentioned, w has finite Dirichlet energy. Taking the trace on Γ, we deduce from the above relation that $q = r \in H(\Gamma)$ and the proof is complete. □

Proposition 2.10 *The operators K_Γ^γ and K_γ^Γ are compact, one-to-one and dense-range operators. Moreover, for every functions $q \in H(\Gamma)$ and $p \in H(\gamma)$, we have:*

$$\langle \mathsf{K}_\Gamma^\gamma q, p\rangle_{\frac12,\gamma} = \langle q, \mathsf{K}_\gamma^\Gamma p\rangle_{\frac12,\Gamma}.$$

Proof : The compactness follows from the regularity of the single layer potential away from the boundary, combined with [44, Proposition 13.5.8].

Addressing the symmetry property, consider $q \in L^\infty(\Gamma) \cap H(\Gamma)$ and $p \in L^\infty(\gamma) \cap H(\gamma)$. We can write that:

$$\begin{aligned}\langle \mathsf{K}_\Gamma^\gamma q, p\rangle_{\frac{1}{2},\gamma} &= \int_\gamma \int_\Gamma G(x-y)\hat{q}(y)\hat{p}(x)\,\mathrm{d}\sigma_y \mathrm{d}\sigma_x \\ &= \int_\Gamma \int_\gamma G(x-y)\hat{q}(x)\hat{p}(y)\,\mathrm{d}\sigma_y \mathrm{d}\sigma_x = \langle \mathsf{K}_\gamma^\Gamma p, q\rangle_{\frac{1}{2},\Gamma},\end{aligned}$$

and the conclusion follows by density.

Assume now that $q \in H(\Gamma)$ is such that $\mathsf{K}_\Gamma^\gamma q = 0$. The function $\mathscr{S}_\Gamma \hat{q}$ is then constant on ω (as it is harmonic in ω and constant on γ). By the unique continuation property for harmonic functions, $\mathscr{S}_\Gamma \hat{q}$ is constant in Ω. On the other hand, $\mathscr{S}_\Gamma \hat{q} \equiv 0$ on $\mathbb{R}^2 \setminus \overline{\Omega}$ (it is harmonic in $\mathbb{R}^2 \setminus \overline{\Omega}$, constant on Γ and it behaves like $O(|x|^{-1})$ at infinity since $\hat{q} \in \hat{H}(\Gamma)$). This yields $\hat{q} = [\partial_n(\mathscr{S}_\Gamma \hat{q})]_\Gamma = 0$ and thus $q = 0$. Since

$$\overline{\operatorname{Ran} \mathsf{K}_\gamma^\Gamma} = \operatorname{Ker} \mathsf{K}_\Gamma^\gamma,$$

we get the density result and the proof is completed. □

Proposition 2.11 *The norms of the operators K_Γ^γ and K_γ^Γ are strictly less that 1.*

Proof : According to Lemma 2.7 we have, for every $q \in H(\Gamma)$:

$$\|\mathsf{K}_\Gamma^\gamma q\|_{\frac{1}{2},\gamma} = \inf\left\{ \|u\|_{W_0^1(\mathbb{R}^2)} \,:\, u \in W_0^1(\mathbb{R}^2) \text{ and } \mathsf{T}r_\gamma^0 u = \mathsf{T}r_\gamma^0(\mathscr{S}_\Gamma \hat{q}) \right\}.$$

We deduce that $\|\mathsf{K}_\Gamma^\gamma q\|_{\frac{1}{2},\gamma} \leqslant \|\mathscr{S}_\Gamma \hat{q}\|_{W_0^1(\mathbb{R}^2)} = \|q\|_{\frac{1}{2},\Gamma}$ and the norm of K_Γ^γ is no greater than 1.

The operator K_Γ^γ being compact, its norm is achieved by some $q_\Gamma \in H(\Gamma)$. If $\|\mathsf{K}_\Gamma^\gamma q_\Gamma\|_{\frac{1}{2},\gamma} = \|q_\Gamma\|_{\frac{1}{2},\Gamma}$, we would have, according to Lemma 2.7:

$$\mathscr{S}_\gamma \hat{q}_\gamma = \mathscr{S}_\Gamma \hat{q}_\Gamma, \qquad \text{in } \mathbb{R}^2,$$

where $q_\gamma := \mathsf{K}_\Gamma^\gamma q_\Gamma$. This identity implies that $\hat{q}_\Gamma = [\partial_n(\mathscr{S}_\gamma \hat{q}_\gamma)]_\Gamma = 0$, yielding the expected contradiction. □

2.3 Integral formulation and well-posedness

Let us go back to Problem (1.2). Without loss of generality, let us assume from now on that the diameter of Ω is less than 1 (otherwise, it suffices to rescale the problem), which implies in particular that $\operatorname{Cap}(\Gamma) < 1$ and $\operatorname{Cap}(\gamma) < 1$ (see [43, p. 143] and references therein).

Proposition 2.12 *For every $f \in H(\Gamma)$, denote by $(u^f, c^f) \in H^1(\Omega \setminus \overline{\omega}) \times \mathbb{R}$ the unique solution of System (1.2). The function u^f can be represented as a superposition of single layer potentials as follows:*

$$u^f = \mathscr{S}_\gamma \hat{p} + \mathscr{S}_\Gamma \hat{q}, \tag{2.4}$$

where $\hat{p} \in \widehat{H}(\gamma)$ and $\hat{q} \in H^{-\frac{1}{2}}(\Gamma)$ solve the following system of coupled integral equations on the boundaries γ and Γ:

$$p + \mathsf{Tr}_\gamma(\mathscr{S}_\Gamma \hat{q}) = c^f \qquad \text{on } \gamma \tag{2.5a}$$

$$\mathsf{Tr}_\Gamma(\mathscr{S}_\gamma \hat{p}) + q = f \qquad \text{on } \Gamma. \tag{2.5b}$$

Proof : It is a consequence of [40, Theorem 8.16] that the unique solution to System (1.2) can be written as a superposition of two single layer potentials respectively supported on Γ and γ and respectively associated with the densities $(\hat{p}, \hat{q}) \in H^{-\frac{1}{2}}(\gamma) \times H^{-\frac{1}{2}}(\Gamma)$, as in (2.4). It only remains to verify that p is in fact in $\widehat{H}(\gamma)$, i.e. that $\langle \hat{p}, 1\rangle_{-\frac{1}{2},\frac{1}{2},\gamma} = 0$. This is a straightforward consequence of (1.2d) and the jump relation for the normal derivative of the single layer potential. $\square$

3 The reconstruction formula

Going back to the DtN operator Λ_γ of problem (1.2), and due to (1.2d), we have by Green's formula

$$\langle \partial_n u^f, 1\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma} = -\langle \partial_n u^f, 1\rangle_{-\frac{1}{2},\frac{1}{2},\gamma} = 0,$$

which shows that Λ_γ is valued in $\widehat{H}(\Gamma)$. Considering data $f \in H(\Gamma)$, we can thus define the DtN operator Λ_γ as follows:

$$\Lambda_\gamma : f \in H(\Gamma) \longmapsto \partial_n u^f \in \widehat{H}(\Gamma). \tag{3.1}$$

In the case where $\omega = \varnothing$, we will denote respectively by u_0^f and Λ_0 the solution u^f and the DtN Λ_γ. Note that we have in particular

$$u_0^f = \mathscr{S}_\Gamma \hat{f}.$$

3.1 Factorization of the DtN map

Theorem 3.1 *The two following bounded linear operators in $H(\Gamma)$:*

$$\mathsf{R} := \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0) \qquad \text{and} \qquad \mathsf{K} := \mathsf{K}_\gamma^\Gamma \mathsf{K}_\Gamma^\gamma,$$

satisfy the following equivalent identities:

$$\mathsf{R} = (\mathrm{Id} - \mathsf{K})^{-1}\mathsf{K}, \qquad\qquad \mathsf{K} = (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}. \tag{3.2}$$

Proof : Given f in $H(\Gamma)$, let $(\hat{p},\hat{q}) \in \widehat{H}(\gamma) \times H^{-\frac{1}{2}}(\Gamma)$ be the solution of System (2.5). According to Lemma 2.9, $\mathsf{Tr}_\Gamma(\mathscr{S}_\gamma \hat{p}) \in H(\Gamma)$ and hence $\mathsf{Tr}_\Gamma(\mathscr{S}_\gamma \hat{p}) = \mathsf{K}_\gamma^\Gamma p$. Since $f \in H(\Gamma)$, we deduce from (2.5b) that $q = f - \mathsf{K}_\gamma^\Gamma p \in H(\Gamma)$. Applying the projector Π_γ to (2.5a), we obtain the following system:

$$p + \mathsf{K}_\Gamma^\gamma q = 0 \qquad \text{on } \gamma \tag{3.3a}$$

$$\mathsf{K}_\gamma^\Gamma p + q = f \qquad \text{on } \Gamma. \tag{3.3b}$$

Eliminating p, it follows that $(\mathrm{Id} - \mathsf{K})q = f$ and hence $(\mathrm{Id} - \mathsf{K})(q - f) = \mathsf{K}f$. The operator K being a contraction (see Propositon 2.11), we end up with:

$$q - f = (\mathrm{Id} - \mathsf{K})^{-1}\mathsf{K}f. \tag{3.4}$$

On the other hand, we have

$$(\Lambda_\gamma - \Lambda_0)f = \partial_n u^f - \partial_n u_0^f, \tag{3.5}$$

where we recall that $u^f = \mathscr{S}_\Gamma \hat{q} + \mathscr{S}_\gamma \hat{p}$ and $u_0^f = \mathscr{S}_\Gamma \hat{f}$. But outside Γ, these two single layer potentials both solve the well-posed Dirichlet exterior boundary value problem:

$$\begin{aligned} -\Delta v &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ v &= f && \text{on } \Gamma, \\ v &= O(|x|^{-1}) && |x| \to +\infty. \end{aligned}$$

Hence $(\mathscr{S}_\Gamma \hat{q} + \mathscr{S}_\gamma \hat{p}) = \mathscr{S}_\Gamma \hat{f}$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, and in particular we can rewrite (3.5) as

$$(\Lambda_\gamma - \Lambda_0)f = [\partial_n(\mathscr{S}_\Gamma \hat{q} + \mathscr{S}_\gamma \hat{p})]_\Gamma - [\partial_n(\mathscr{S}_\Gamma \hat{f})]_\Gamma = \hat{q} - \hat{f},$$

where the last equality follows from the jump relation (2.2). Comparing this relation and (3.4), we obtain that $\mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0)f = (\mathrm{Id} - \mathsf{K})^{-1}\mathsf{K}f$, which is exactly the first relation in (3.2). The second relation follows easily. $\square$

The first equation in (3.2) can be seen as a factorization of the (known) DtN operator $\Lambda_\gamma - \Lambda_0$ in terms of the (unknown) boundary interaction operator K_Γ^γ and K_γ^Γ. Similarly, the second equation in (3.2) can be seen as a factorization of the boundary interaction operator $\mathsf{K} = \mathsf{K}_\gamma^\Gamma \mathsf{K}_\Gamma^\gamma$ in terms of the measurement operator R (which is entirely determined by the perturbed and unperturbed DtN maps and by the exterior boundary Γ). Using Proposition 2.10, it is worth reformulating this second equation in a variational form:

$$\langle \mathsf{K}_\Gamma^\gamma f, \mathsf{K}_\Gamma^\gamma g \rangle_{\frac{1}{2},\gamma} = \langle (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}f, g \rangle_{\frac{1}{2},\Gamma}, \qquad \forall\, f, g \in H(\Gamma). \tag{3.6}$$

This identity constitutes the first step towards the reconstruction of the unknown boundary γ. Indeed, the bilinear form $\langle \mathsf{K}_\Gamma^\gamma \cdot, \mathsf{K}_\Gamma^\gamma \cdot \rangle_{\frac{1}{2},\gamma}$ turns out to encode the geometry of the inclusion, as shown in the next section.

3.2 Harmonic polynomials and GPST

Throughout the paper, we identify $x = (x_1, x_2)$ in $\mathbb{R}^2$ with the complex number $z = x_1 + ix_2$.

Definition 3.2 *For every $m \geqslant 1$, we define the harmonic polynomials of degree m:*

$$P_1^m(x) = \mathrm{Re}\,(z^m) \qquad \textit{and} \qquad P_2^m(x) = \mathrm{Im}\,(z^m)\,.$$

We define as well

$$Q_{1,\Gamma}^m(x) := P_1^m(x) + c_{1,\Gamma}^m \qquad \textit{and} \qquad Q_{2,\Gamma}^m(x) := P_2^m(x) + c_{2,\Gamma}^m \tag{3.7}$$

where the constant $c_{\ell,\Gamma}^m \in \mathbb{R}$, $\ell = 1, 2$, are chosen such that the trace of $Q_{\ell,\Gamma}^m$ on Γ belongs to $H(\Gamma)$.
Finally, we set

$$Q_\Gamma^m := Q_{1,\Gamma}^m + iQ_{2,\Gamma}^m. \tag{3.8}$$

The crucial point about these polynomials $Q_{\ell,\Gamma}^m$, $\ell = 1, 2$, lies in the fact that since they are harmonic, we have

$$\mathsf{K}_\Gamma^\gamma(Q_{\ell,\Gamma}^m\big|_\Gamma) = Q_{\ell,\gamma}^m\big|_\gamma,$$

and hence, using these harmonic polynomials $Q_{\ell,\Gamma}^m$ in formula (3.6) (and using for simplicity the same notation for the functions and their traces on the boundaries γ and Γ), we obtain that for all $m, m' \geqslant 1$ and all $\ell, \ell' = 1, 2$:

$$\langle \mathsf{K}_\Gamma^\gamma Q_{\ell,\Gamma}^m, \mathsf{K}_\Gamma^\gamma Q_{\ell',\Gamma}^{m'} \rangle_{\frac{1}{2},\gamma} = \langle Q_{\ell,\gamma}^m, Q_{\ell',\gamma}^{m'} \rangle_{\frac{1}{2},\gamma}.$$

Remark 3.3 *The quantities $\langle Q_{\ell,\gamma}^m, Q_{\ell',\gamma}^{m'} \rangle_{\frac{1}{2},\gamma}$ are strongly connected with the so-called Generalized Pólya-Szegö Tensors (GPST) appearing in the high-order asymptotic expansion of the DtN map for small inclusions (see for instance the recent papers by Ammari et al. [5, 4] and references therein). Our definition is somehow different from theirs, as they use real polynomials x^m, while we use harmonic polynomials.*

3.3 From the GPST to the geometry of the cavity: an explicit inversion formula

In this section, we are going to see that the quantities $\langle Q_{\ell,\gamma}^m, Q_{\ell',\gamma}^{m'} \rangle_{\frac{1}{2},\gamma}$ for $m, m' \geqslant 1$ and $\ell, \ell' = 1, 2$, which can be deduced from the measurements (see (3.6)), contain all the necessary information to reconstruct the cavity. We can even say more: the geometric information of γ is actually redundant in the GPST. As we shall see, the knowledge of the quantities $\langle Q_\gamma^m, Q_\gamma^1 \rangle_{\frac{1}{2},\gamma}$ and

$\langle Q_\gamma^m, \overline{Q_\gamma^1}\rangle_{\frac{1}{2},\gamma}$ suffices to reconstruct the cavity. More precisely, according to Pommerenke [41, p. 5], there exists a mapping

$$\phi : z \mapsto a_1 z + a_0 + \sum_{m\leqslant -1} a_m z^m,$$

that maps the exterior of the unit disk D onto the exterior of ω (the assumption that ω is bounded and simply connected is crucial here). In particular, $t \in]-\pi,\pi] \mapsto \phi(e^{it})$ provides a parameterization of γ. Notice that in this description, $|a_1|$ is the logarithmic capacity of γ and a_1 can be chosen as a positive real. The coefficient a_0 is the conformal center of ω. With these notation, we have the following result.

Theorem 3.4 *Let $(Q_\gamma^m)_{m\geqslant 1}$ be the complex harmonic polynomials defined by* (3.7)-(3.8)*. Define the two following sequences of complex numbers ($1 \leqslant m \leqslant +\infty$):*

$$\mu_m := \frac{1}{2}\langle Q_\gamma^m, \overline{Q_\gamma^1}\rangle_{\frac{1}{2},\gamma} = \frac{1}{2}\langle Q_\Gamma^m, (\mathrm{Id}+\mathsf{R})^{-1}\mathsf{R}\overline{Q_\Gamma^1}\rangle_{\frac{1}{2},\Gamma}, \tag{3.9a}$$

$$\nu_m := \frac{1}{2}\langle Q_\gamma^m, Q_\gamma^1\rangle_{\frac{1}{2},\gamma} = \frac{1}{2}\langle Q_\Gamma^m, (\mathrm{Id}+\mathsf{R})^{-1}\mathsf{R}Q_\Gamma^1\rangle_{\frac{1}{2},\Gamma}, \tag{3.9b}$$

with $\mathsf{R} := \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0)$*. Then, $\mu_1 > 0$ and we have the explicit formulae:*

$$a_1 = \left(\frac{\mu_1}{2\pi}\right)^{\frac{1}{2}} \qquad\qquad a_0 = \frac{\mu_2}{2\mu_1} \tag{3.9c}$$

$$a_{-m} = \mu_1^{-\frac{m}{2}} \sum_{\alpha\in\mathsf{A}_m} C_\alpha \left(\frac{\mu_2}{\mu_1}\right)^{\alpha_0} \nu_1^{\alpha_1}\nu_2^{\alpha_2}\ldots\nu_m^{\alpha_m}, \quad m \geqslant 1, \tag{3.9d}$$

where

$$\mathsf{A}_m := \{\alpha \in \mathbb{N}^{m+1} \,:\, \alpha_0 + 2\alpha_1 + 3\alpha_2 + \ldots + (m+1)\alpha_m = (m+1),\ \alpha_0 \neq m+1\} \tag{3.9e}$$

and

$$C_\alpha := \frac{(-1)^{|\alpha|+1}}{2^{\alpha_0} m} \frac{(2\pi)^{\frac{m}{2}-(\alpha_1+\cdots+\alpha_m)}}{1^{\alpha_1}2^{\alpha_2}\ldots m^{\alpha_m}}. \tag{3.9f}$$

The rest of this section is devoted to the proof of this result.

To simplify the forthcoming computation, we complete the sequence of complex numbers $(a_k)_{k\leqslant 1}$ by setting $a_k = 0$ for $k \geqslant 2$. We denote a_k^n ($n \in \mathbb{N}$, $k \in \mathbb{Z}$) the kth coefficients of the Laurent's series of ϕ^n:

$$a_k^n = \sum_{|\alpha|=k} a_{\alpha_1}a_{\alpha_2}\ldots a_{\alpha_n}, \tag{3.10}$$

where the sum ranges over all the multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$ whose length $|\alpha| = \alpha_1 + \ldots + \alpha_n$ is equal to k. We also introduce the quantities:

$$\mu^{m,m'} := \frac{1}{2}\langle Q_\gamma^m, \overline{Q_\gamma^{m'}}\rangle_{\frac{1}{2},\gamma} = \frac{1}{2}\langle Q_\Gamma^m, (\mathrm{Id}+\mathsf{R})^{-1}\mathsf{R}\overline{Q_\Gamma^{m'}}\rangle_{\frac{1}{2},\Gamma}$$

$$\nu^{m,m'} := \frac{1}{2}\langle Q^m_\gamma, Q^{m'}_\gamma\rangle_{\frac12,\gamma} = \frac{1}{2}\langle Q^m_\Gamma, (\mathrm{Id}+\mathsf{R})^{-1}\mathsf{R}Q^{m'}_\Gamma\rangle_{\frac12,\Gamma},$$

so that

$$\mu_m = \mu^{m,1} \qquad \text{and} \qquad \nu_m = \nu^{m,1}.$$

Lemma 3.5 *Denoting, for every $m \geqslant 1$:*

$$\phi^m_+(z) = \sum_{k\geqslant 1} a^m_k z^k \qquad \textit{and} \qquad \phi^m_-(z) = \sum_{k\leqslant -1} a^m_k z^k,$$

the following identities hold true:

$$\mu^{m,m'} = \int_{-\pi}^{\pi} \overline{e^{it}(\phi^{m'}_+)'(e^{it})}\phi^m(e^{it})\,\mathrm{d}t \tag{3.11a}$$

$$\nu^{m,m'} = \int_{-\pi}^{\pi} e^{it}(\phi^{m'}_+)'(e^{it})\phi^m(e^{it})\,\mathrm{d}t. \tag{3.11b}$$

Proof : Let $m, m' \geqslant 1$ and $\ell = 1, 2$ be fixed. For the sake of simplicity, we drop in this proof the dependence with respect to γ and we denote $Q^m_{\ell,\gamma}$ simply by Q^m_ℓ. We aim to compute the quantity:

$$\langle Q^m_\ell, Q^{m'}_\ell\rangle_{\frac12,\gamma} = \langle \widehat{Q}^m_\ell, Q^{m'}_\ell\rangle_{-\frac12,\frac12,\gamma}.$$

To do so, we recall that from the jump relation (2.2), we have $\widehat{Q}^m_\ell = [\partial_n U^m_\ell]_\gamma$, where $U^m_\ell := \mathscr{S}_\gamma \widehat{Q}^m_\ell$. Let us denote by $U^m_{e,\ell}$ and $U^m_{i,\ell}$ the restrictions of U^m_ℓ respectively to $\mathbb{R}^2 \setminus \overline{\omega}$ and ω.

We know that $U^m_{e,\ell}$ solves the following exterior Dirichlet boundary problem:

$$-\Delta U^m_{e,\ell} = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{\omega} \tag{3.12a}$$

$$U^m_{e,\ell} = Q^m_\ell \qquad \text{on } \gamma, \tag{3.12b}$$

$$U^m_{e,\ell}(x) = O(|x|^{-1}) \qquad \text{as } |x| \to +\infty. \tag{3.12c}$$

The functions $u^m_{e,\ell} := U^m_{e,\ell}(\phi)$ are harmonic in $\mathbb{R}^2 \setminus \overline{D}$ (D denotes the unit disk) and satisfy:

$$u^m_{e,1}(x) = \mathrm{Re}\,(\phi^m(z)) + c^m_1 \qquad \text{and} \qquad u^m_{e,2}(x) = \mathrm{Im}\,(\phi^m(z)) + c^m_2.$$

We can easily compute the constants c^m_1 and c^m_2 by writing that

$$c^m_1 + ic^m_2 = -\int_\gamma (P^m_1(x) + i\,P^m_2(x))\hat{\mathsf{e}}_\gamma(x)\,\mathrm{d}\sigma_x = -\int_{-\pi}^{\pi} \phi^m(e^{it})|\phi'(e^{it})|\hat{\mathsf{e}}_\gamma(e^{it})\,\mathrm{d}t.$$

But we know (from direct computations or from [27, Theorem 17.3.3]) that $\hat{\mathsf{e}}_\gamma(e^{it}) = 1/(2\pi|\phi'(e^{it})|)$. It follows that $c_1^m + ic_2^m = -a_0^m$ and we have, on the boundary of D:

$$u_{e,1}^m(x) = \frac{1}{2}\left[\phi^m(z) + \overline{\phi^m(z)}\right] - \mathrm{Re}\,(a_0^m)$$
$$u_{e,2}^m(x) = -\frac{i}{2}\left[\phi^m(z) - \overline{\phi^m(z)}\right] - \mathrm{Im}\,(a_0^m).$$

This can be rewritten, using the identity $\bar{z} = 1/z$ on ∂D, as:

$$u_{e,1}^m(x) = \frac{1}{2}\left[\overline{\phi_+^m}(z^{-1}) + \phi_-^m(z) + \overline{\overline{\phi_+^m}(z^{-1}) + \phi_-^m(z)}\right] \tag{3.13a}$$
$$u_{e,2}^m(x) = -\frac{i}{2}\left[-\overline{\phi_+^m}(z^{-1}) + \phi_-^m(z) - \left(\overline{-\overline{\phi_+^m}(z^{-1}) + \phi_-^m(z)}\right)\right]. \tag{3.13b}$$

These expressions lead us to introduce the following functions:

$$w_1^m(z) = \overline{\phi_+^m}(z^{-1}) + \phi_-^m(z) = \phi^m(z) - a_0^m + \lambda_1^m(z) \tag{3.14a}$$
$$w_2^m(z) = -\overline{\phi_+^m}(z^{-1}) + \phi_-^m(z) = \phi^m(z) - a_0^m + \lambda_2^m(z), \tag{3.14b}$$

where

$$\lambda_1^m(z) = \overline{\phi_+^m}(z^{-1}) - \phi_+^m(z) \qquad \text{and} \qquad \lambda_2^m(z) = -\overline{\phi_+^m}(z^{-1}) - \phi_+^m(z). \tag{3.15}$$

The functions w_1^m and w_2^m are holomorphic in $\mathbb{C} \setminus \overline{D}$ and:

$$u_{e,1}^m = \mathrm{Re}\,(w_1^m) \quad \text{and} \quad u_{e,2}^m = \mathrm{Im}\,(w_2^m) \quad \text{on } \partial D.$$

For every $X = (X_1, X_2) \in \mathbb{R}^2$ identified with $Z = X_1 + iX_2 \in \mathbb{C}$, we have:

$$\nabla U_{e,1}^m(\phi(z)) \cdot X = \mathrm{Re}\,\left[(w_1^m)'(z)Z/\phi'(z)\right]$$
$$\nabla U_{e,2}^m(\phi(z)) \cdot X = \mathrm{Im}\,\left[(w_2^m)'(z)Z/\phi'(z)\right].$$

On γ, the outer unit normal vector is parameterized by

$$t \in [-\pi, \pi[\longmapsto e^{it}\phi'(e^{it})/|\phi'(e^{it})|,$$

and therefore, for every $m' \geqslant 1$:

$$\int_\gamma \partial_n U_{e,1}^m(x)\, Q_1^{m'}(x)\mathrm{d}\sigma_x = \int_{-\pi}^{\pi} \mathrm{Re}\,\left[e^{it}(w_1^m)'(e^{it})\right]\, \mathrm{Re}\left[\phi^{m'}(e^{it})\right]\mathrm{d}t. \tag{3.16}$$

On the other hand, $U_{i,\ell}^m$ solves the following interior problem:

$$-\Delta U_{i,\ell}^m = 0 \qquad \text{in } \omega$$
$$U_{i,\ell}^m = Q_\ell^m \quad \text{on } \gamma,$$

whose unique solution is merely $U^m_{i,\ell} = Q^m_\ell$, so that:

$$\int_\gamma \partial_n U^m_{i,1}(x) Q^{m'}_1(x)\mathrm{d}\sigma_x = \int_{-\pi}^{\pi} \mathrm{Re}\,\left[e^{it}(\phi^m)'(e^{it})\right] \mathrm{Re}\left[\phi^{m'}(e^{it})\right]\mathrm{d}t. \tag{3.17}$$

Gathering now (3.16), (3.17) and taking into account the expressions (3.14), we infer that:

$$\langle Q^m_1, Q^{m'}_1\rangle_{\frac{1}{2},\gamma} = \int_\gamma \left[\partial_n U^m_1(x)\right]_\gamma Q^{m'}_1(x)\mathrm{d}\sigma_x \tag{3.18a}$$

$$= -\int_{-\pi}^{\pi} \mathrm{Re}\,\left[e^{it}(\lambda^m_1)'(e^{it})\right] \mathrm{Re}\left[\phi^{m'}(e^{it})\right]\mathrm{d}t. \tag{3.18b}$$

Notice now that, on ∂D, we have:

$$\mathrm{Re}\,\left[e^{it}(\lambda^m_1)'(e^{it})\right] = -\left[\overline{e^{it}(\phi^m_+)'(e^{it})} + e^{it}(\phi^m_+)'(e^{it})\right] = -2\mathrm{Re}\,\left[e^{it}(\phi^m_+)'(e^{it})\right],$$

and therefore, (3.18) can be rewritten as:

$$\langle Q^m_1, Q^{m'}_1\rangle_{\frac{1}{2},\gamma} = 2\int_{-\pi}^{\pi} \mathrm{Re}\,\left[e^{it}(\phi^m_+)'(e^{it})\right] \mathrm{Re}\,\left[\phi^{m'}(e^{it})\right]\mathrm{d}t.$$

Using similar arguments, lengthy but straightforward computations lead to:

$$\begin{aligned}\langle Q^m_1, Q^{m'}_2\rangle_{\frac{1}{2},\gamma} &= 2\int_{-\pi}^{\pi} \mathrm{Re}\,\left[e^{it}(\phi^m_+)'(e^{it})\right] \mathrm{Im}\,\left[\phi^{m'}(e^{it})\right]\mathrm{d}t \\ &= 2\int_{-\pi}^{\pi} \mathrm{Im}\,\left[e^{it}(\phi^m_+)'(e^{it})\right] \mathrm{Re}\,\left[\phi^{m'}(e^{it})\right]\mathrm{d}t\end{aligned}$$

and

$$\langle Q^m_2, Q^{m'}_2\rangle_{\frac{1}{2},\gamma} = 2\int_{-\pi}^{\pi} \mathrm{Im}\,\left[e^{it}(\phi^m_+)'(e^{it})\right] \mathrm{Im}\,\left[\phi^{m'}(e^{it})\right]\mathrm{d}t.$$

Formulae (3.11) follow. □

Lemma 3.6 *The following relations hold true:*

$$a_1 = \sqrt{\frac{\mu_1}{2\pi}} \quad \textit{and} \quad a_0 = \frac{\mu_2}{2\mu_1}.$$

Proof : For every $z \in \mathbb{C} \setminus \overline{D}$ (recall that D denotes the unit disk), we have $\phi^1_+(z) = \phi_+(z) = a_1 z$ and $\phi^2_+(z) = (a_1)^2 z^2 + 2a_1 a_0 z$, and hence

$$z\phi'_+(z) = a_1 z \qquad \text{and} \qquad z(\phi^2_+)'(z) = 2(a_1)^2 z^2 + 2a_1 a_0.$$

Applying formulae (3.11), we obtain:

$$\mu_1 = 2\pi(a_1)^2 \qquad \text{and} \qquad \mu_2 = 4\pi(a_1)^2 a_0.$$

The conclusion of the lemma follows. □

The conformal mapping ϕ^{-1} can be expanded as follows:

$$\phi^{-1}(z) = b_1 z + b_0 + \sum_{k\leqslant -1} b_k z^k,$$

outside a disk D' centered at the origin and containing ω. This follows from the fact that $\phi^{-1}(z)$ admits a Laurent's series expansion as being the inverse of a holomorphic function. Moreover, the series has no terms of degree $k > 1$ since ϕ (and thus ϕ^{-1}) behaves like $O(|z|)$ at infinity. The complex coefficients b_k ($k \leqslant 1$) can be deduced on the one hand from the coefficients a_k of ϕ, and on the other hand from the values of ν_m, ($m \geqslant 1$), as claimed in the following lemma:

Lemma 3.7 *The following relations hold true:*

$$b_1 = 1/a_1 \quad \textit{and} \quad b_0 = -a_0/a_1. \tag{3.19}$$

For every $m \geqslant 1$, we have:

$$b_{-m} = -\frac{\nu_m}{2\pi a_1 m} = -\frac{1}{m}\sum_{|\beta|=-1} a_{\beta_1}\dots a_{\beta_m}, \qquad m \geqslant 1. \tag{3.20}$$

Proof : Identities (3.19) follow straightforwardly because ϕ and ϕ^{-1} are inverse mappings. Integrating by part the expression of $\nu_m = \nu^{m,1}$ obtained from (3.11), we get:

$$\nu_m = a_1 \int_{-\pi}^{\pi} e^{it}\phi^m(e^{it})\,\mathrm{d}t = -a_1 m \int_{-\pi}^{\pi} e^{it}\phi'(e^{it})e^{it}\phi^{m-1}(e^{it})\,\mathrm{d}t. \tag{3.21}$$

Since $t \mapsto e^{it}$ is a parameterization of ∂D, applying Cauchy's integral formula we get on the one hand

$$\int_{-\pi}^{\pi} e^{it}\phi^m(e^{it})\,\mathrm{d}t = -i\int_{\partial D}\phi^m(\xi)\mathrm{d}\xi = 2\pi a_{-1}^m.$$

On the other hand, since $t \mapsto \phi(e^{it})$ is a parameterization of γ and the function ϕ^{-1} being holomorphic in $\mathbb{C}\setminus\overline{\omega}$, we have:

$$\begin{aligned}\int_{-\pi}^{\pi} e^{it}\phi'(e^{it})e^{it}\phi^{m-1}(e^{it})\,\mathrm{d}t &= -i\int_{\gamma}\phi^{-1}(\xi)\xi^{m-1}\,\mathrm{d}\xi \\ &= -i\int_{\partial D'}\phi^{-1}(\xi)\xi^{m-1}\,\mathrm{d}\xi = 2\pi b_{-m}.\end{aligned}$$

Identity (3.21) can thus be rewritten as:

$$\nu_m = 2\pi a_1 a_{-1}^m = -2\pi a_1 m b_{-m},$$

and identity (3.20) follows according to (3.10). $\square$

Using the above lemmas, we are in position to prove the main result of this section, namely Theorem 3.4.

Proof of Theorem 3.4: Since ϕ and ϕ^{-1} play symmetric roles, we can exchange a_m and b_m in Formula (3.20) to obtain:

$$a_{-m} = -\frac{1}{m} \sum_{|\beta|=-1} b_{\beta_1} \dots b_{\beta_m}, \qquad m \geqslant 1.$$

Reordering the terms of the above sum, we get that

$$a_{-m} = -\frac{1}{m} \sum_{(\theta,\alpha)\in \mathsf{B}_m} b_1^{\theta} b_0^{\alpha_0} b_{-1}^{\alpha_1} \dots b_{-m}^{\alpha_m} \tag{3.22}$$

where B_m is the set of $(\theta, \alpha) \in \mathbb{N} \times \mathbb{N}^{m+1}$ such that

$$\theta + \alpha_0 + \alpha_1 + \dots + \alpha_m = m$$
$$\theta - (\alpha_1 + 2\alpha_2 + \dots + m\alpha_m) = -1.$$

Now, one can easily check that $(\theta, \alpha) \in \mathsf{B}_m$ if and only if α belongs to the set A_m defined by (3.9e) and $\theta = m - (\alpha_0 + \alpha_1 + \dots + \alpha_m)$. Therefore, (3.22) also reads

$$a_{-m} = -\frac{1}{m} \sum_{\alpha \in \mathsf{A}_m} b_1^{m-(\alpha_0+\alpha_1+\dots+\alpha_m)} b_0^{\alpha_0} b_{-1}^{\alpha_1} \dots b_{-m}^{\alpha_m}$$

Using (3.19) and the first equality of (3.20) in the above relation, we obtain that

$$a_{-m} = -\frac{1}{m} a_1^{-m} \sum_{\alpha \in \mathsf{A}_m} (-a_0)^{\alpha_0} \left(-\frac{\nu_1}{2\pi}\right)^{\alpha_1} \dots \left(-\frac{\nu_m}{2\pi m}\right)^{\alpha_m},$$

and the conclusion follows immediately. $\square$

3.4 About stability

It is well-known that logarithmic stability is best possible for Calderón's inverse problem. In the particular case of cavities, this result is proved in [2, Theorem 4.1] where the error on the geometry (measured using the Hausdorff distance) is estimated in terms of the error of the DtN (measured in operator norm). However, as suggested by Alessandrini and Vessella [3], one can try to construct stable functionals, namely Lipschitz-continuous functions of the data carrying relevant information on the geometry of the obstacle. According to formula (3.9), each coefficient a_k, $k \leqslant 1$, yields an example of such functional. Actually, we can prove that each coefficient is not only a Lipschitz-continuous function of the data, but is analytic. Let us define the following open subspace of $\mathcal{L}(H(\Gamma))$:

$$\mathcal{U}_\Gamma = \{\mathsf{R} \in \mathcal{L}(H(\Gamma)) \,:\, \mathrm{Id} + \mathsf{R} \text{ invertible and } \mu_1(\mathsf{R}) > 0\},$$

where $\mu_1(\mathsf{R}) := \langle Q^1_\Gamma, (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}\overline{Q^1_\Gamma}\rangle_{\frac{1}{2},\Gamma}$.

Notice in particular that, for every Lipschitz Jordan curve γ, the continuous linear mapping $\mathsf{R} = \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0)$ belongs to $\mathcal{U}_\Gamma$. We deduce straightforwardly the following analyticity result:

Theorem 3.8 *On the open subset $\mathcal{U}_\Gamma$ of $\mathcal{L}(H(\Gamma))$ define the sequence of analytic functions $a_k : \mathcal{U}_\Gamma \to \mathbb{C}$ $(k \leqslant 1)$ as given by the formulae* (3.9). *If $\mathsf{R} = \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0)$ for some Lipschitz Jordan curve γ, a parameterization of γ is given by:*

$$t \in]-\pi, \pi] \mapsto \sum_{k \leqslant 1} a_k(\mathsf{R}) e^{ikt} \in \mathbb{C}.$$

At least theoretically, each coefficient a_k can be computed in a stable way from a given noisy data using formulae (3.9). However, one cannot expect to recover in a stable way the whole sequence $(a_k)_{k\leqslant 1}$ (and thus the geometry of the cavity). In practice, the level of noise limits the number of coefficients that can be accurately recovered.

4 Numerical results

We present in this section some numerical experiments meant to illustrate the feasibility of the proposed reconstruction method. For the sake of clarity, let us first sum up the different steps of the simple reconstruction algorithm:

1. Compute a numerical approximation of the operator $\mathsf{R} = \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0)$.

2. Fix an integer M and compute for $1 \leqslant m \leqslant M$

$$\mu_m := \frac{1}{2}\langle Q^m_\gamma, \overline{Q^1_\gamma}\rangle_{\frac{1}{2},\gamma} = \frac{1}{2}\langle Q^m_\Gamma, (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}\overline{Q^1_\Gamma}\rangle_{\frac{1}{2},\Gamma}$$

$$\nu_m := \frac{1}{2}\langle Q^m_\gamma, Q^1_\gamma\rangle_{\frac{1}{2},\gamma} = \frac{1}{2}\langle Q^m_\Gamma, (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}Q^1_\Gamma\rangle_{\frac{1}{2},\Gamma}.$$

3. Compute $(a_{-m})_{-1\leqslant m\leqslant M}$ via formulae (3.9).

4. Plot the image of the unit circle by

$$\phi_M(z) = a_1 z + a_0 + \sum_{1\leqslant m\leqslant M} a_{-m} z^{-m}.$$

Let us give some details about the implementation. We use the finite dimensional approximation space spanned by the family $\mathcal{Q}_\Gamma^M := \{Q_\Gamma^m, \overline{Q_\Gamma^m}, 1 \leqslant m \leqslant M\}$. We denote by $\mathbf{Q}_\Gamma$ the $2M \times 2M$ complex matrix whose entries are the $\langle f, g\rangle_{\frac{1}{2},\Gamma}$, where $(f,g) \in \mathcal{Q}_\Gamma^M \times \mathcal{Q}_\Gamma^M$. Note that $\mathbf{Q}_\Gamma$ is nothing but the Generalized Polya-Szegö Tensor (GPST) associated with Γ. Obviously, a similar matrix $\mathbf{Q}_\gamma$ can be defined for the boundary γ. We denote by $\mathbf{R}$ the matrix whose entries are $\langle f, \mathsf{R}g\rangle_{\frac{1}{2},\Gamma} = \langle(\Lambda_\gamma - \Lambda_0)g, f\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma}$, for $(f,g) \in \mathcal{Q}_\Gamma^M \times \mathcal{Q}_\Gamma^M$. With this notation, the reader can easily check that formula (3.6) admits the following discrete version:

$$\mathbf{Q}_\gamma \simeq \widetilde{\mathbf{Q}}_\gamma := \mathbf{Q}_\Gamma(\mathbf{Q}_\Gamma + \mathbf{R})^{-1}\mathbf{R}. \tag{4.1}$$

This formula relates in a very simple way, through the measurement operator $\mathbf{R}$, the GPST of γ to the GPST of Γ. In particular, the coefficients μ_m and ν_m are particular entries of $\frac{1}{2}\mathbf{Q}_\gamma$.

We consider now a test configuration in which Γ is an ellipse centered at the origin and of major axis $[-1.9, 1.9]$ and minor axis $[-1.1, 1.1]$. The boundary γ of the obstacle is parameterized by:

$$t \in]-\pi, \pi] \mapsto \sum_{k=-7}^{1} a_k e^{ikt},$$

where the complex coefficients a_k are given in the following table:

a_1	a_0	a_{-1}	a_{-2}	a_{-3}	a_{-4}	a_{-5}	a_{-6}	a_{-7}
0.5	-1	0.085	$-0.06i$	-0.035	$0.06i$	0	$-0.01i$	-0.005

The data are generated using the `Matlab` Laplace boundary integral equation solver (for more information, see this link: IES). Taking $M = 12$, we first show on Figure 2 the reconstructed cavity for exact data and using the eight coefficients $a_1, \ldots, a_{-6}$.

Instead of using the harmonic polynomials z^n in Definition 3.2, one can use the shifted harmonic polynomials $(z-r)^n$, for some given $r \in \mathbb{C}$. This additional parameter turns out to have some influence on the quality of the reconstructed cavities, as shown in Figure 3. For instance, choosing r in the neighborhood of -0.5, one can recover the six coefficients $a_1, \ldots, a_{-4}$ with a relative error less than 2%, while for $r = 0$ this accuracy is achieved only for the coefficients $a_1, \ldots, a_{-2}$.

Let us take now $r = -0.5$ and consider a more realistic configuration of noisy data. We generate a random matrix $\mathbf{N}$ having the same size as

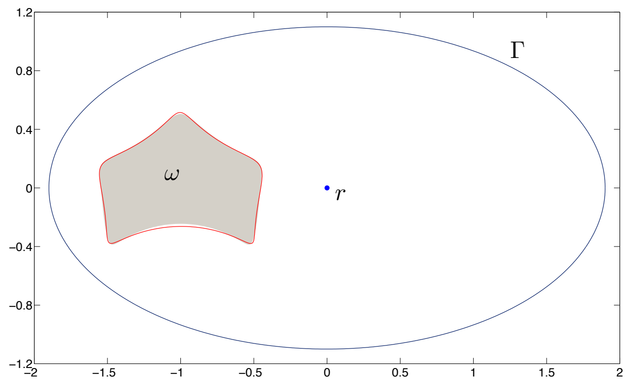

Figure 2: Typical configuration: reconstruction with $a_1, \ldots, a_{-6}$ (in red) and actual inclusion (in gray). The blue point stands for the position of the origin r.

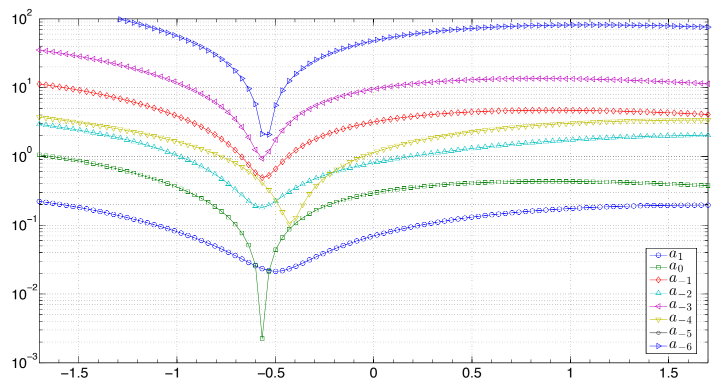

Figure 3: Relative error of the retrieved coefficients (in %) with respect to the abscissa of r (there is no relative error for the coefficient a_{-5} because it is null).

$\mathbf{R}$ and whose coefficients are uniformly distributed between -1 and 1. For $\delta = 0.05, 0.15, 0.25, 0.35$, we compute the matrix $\mathbf{R}^N$ whose coefficients are:

$$R_{ij}^N = (1 + \delta N_{ij}) R_{ij}, \qquad 1 \leqslant i, j \leqslant 2M,$$

and we replace $\mathbf{R}$ by $\mathbf{R}^N$ in formula (4.1).

We show on figures 4-7 examples of reconstructed cavities respectively with $5\%, 15\%, 25\%$ and 35% of noise. The number of correctly recovered coefficients decreases with the level of noise and only those coefficients are used in the reconstruction. We plot on Figure 8 the dependence of the mean relative error with respect to the level of noise and we notice a good stability of the first three coefficients a_1, a_0 and a_{-1}.

Finally, we illustrate on Figure 9 the efficiency of the method for more complex geometries (non convex outer boundary Γ and a non centered cavity).

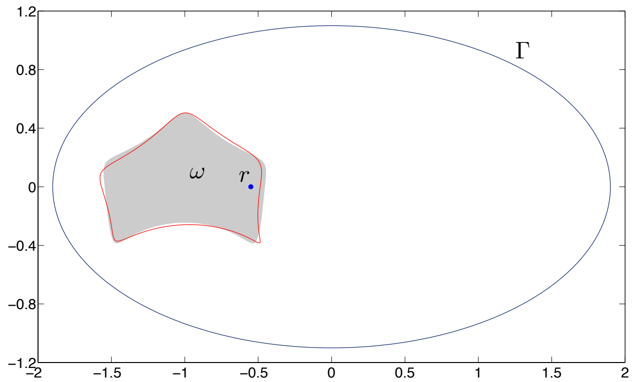

Figure 4: Reconstruction (in red) with $a_1, \ldots, a_{-4}$ and actual inclusion (in gray) with 5% noise.

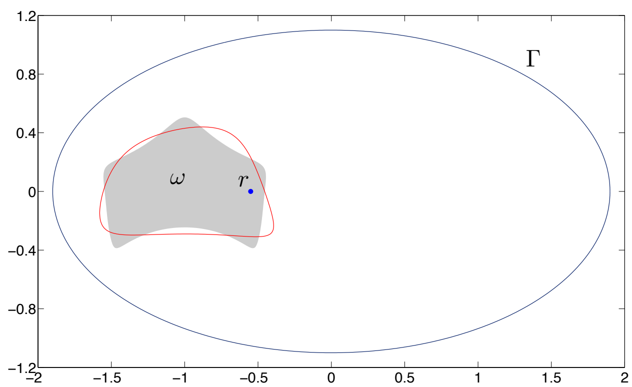

Figure 5: Reconstruction (in red) with $a_1, \ldots, a_{-4}$ and actual inclusion (in gray) with 15% noise.

The choice of the parameter r to obtain good reconstructions is not clear so far and this would need to be further investigated.

A Appendix

Consider problem (1.1) with a piecewise conductivity $\sigma(x) = 1 + (\alpha - 1)1_\omega(x)$ (where 1_ω denotes the characteristic function of ω and α a positive constant).

Proposition A.1 *For every $f \in H^{\frac{1}{2}}(\Gamma)$, System (1.2) admits a unique solution $(u^f, c^f) \in H^1(\Omega \setminus \overline{\omega}) \times \mathbb{R}$. It is the unique pair that realizes:*

$$\min_{(u,c)\in H^1(\Omega\setminus\overline{\omega})\times\mathbb{R}} \left\{ \frac{1}{2} \int_{\Omega\setminus\overline{\omega}} |\nabla u|^2 \mathrm{d}x \,:\, u|_\Gamma = f,\, u|_\gamma = c \right\}. \tag{A.1}$$

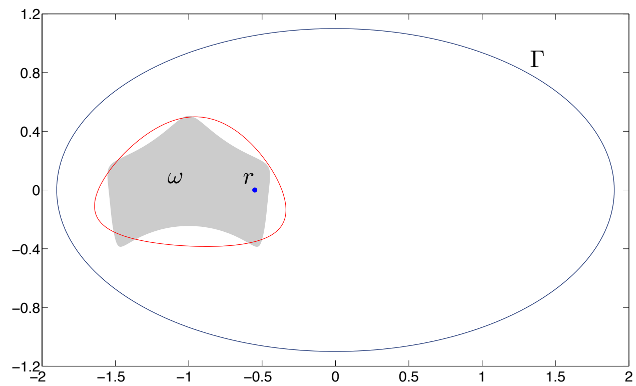

Figure 6: Reconstruction (in red) with $a_1, \ldots, a_{-2}$ and actual inclusion (in gray) with 25% noise.

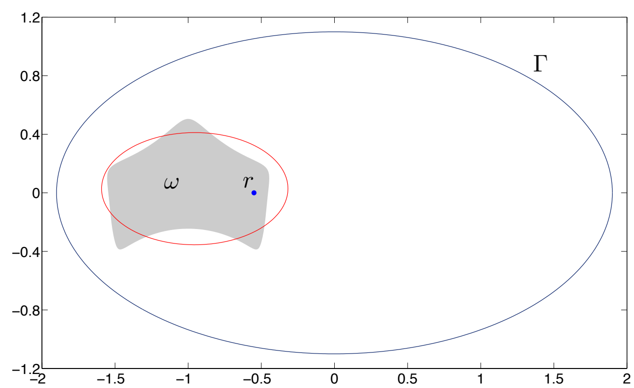

Figure 7: Reconstruction (in red) with $a_1, \ldots, a_{-1}$ and actual inclusion (in gray) with 35% noise.

The function u^f can be considered as a function of $H^1(\Omega)$ by setting $u^f = c^f$ in ω.

For every $\alpha > 0$, System (1.1) admits a unique solution $u^f_\alpha \in H^1(\Omega)$. This function achieves:

$$\min_{u \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega \setminus \overline{\omega}} |\nabla u|^2 \mathrm{d}x + \frac{\alpha}{2} \int_\omega |\nabla u|^2 \mathrm{d}x \,:\, u|_\Gamma = f \right\}. \tag{A.2}$$

The following convergence result holds true for every $f \in H^{\frac{1}{2}}(\Gamma)$:

$$u^f_\alpha \to u^f \qquad \text{in } H^1(\Omega) \text{ as } \alpha \to +\infty.$$

Proof : The minimization problem (A.1) can be reformulated as:

$$\min_{(w,c) \in H^1_0(\Omega \setminus \overline{\omega}) \times \mathbb{R}} \int_{\Omega \setminus \overline{\omega}} |\nabla(w + e^f + cv)|^2 \mathrm{d}x, \tag{A.3}$$

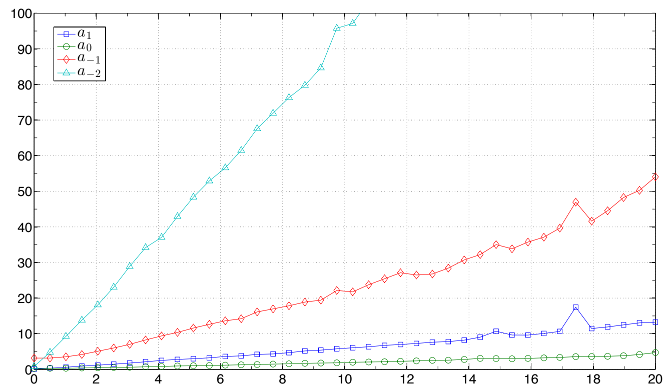

Figure 8: Relative error of the retrieved coefficients (in %) with respect to the level of noise (in %) of the data. The point r is the center of the ellipse. Notice the stability of the conformal center a_0 and the logarithmic capacity a_1.

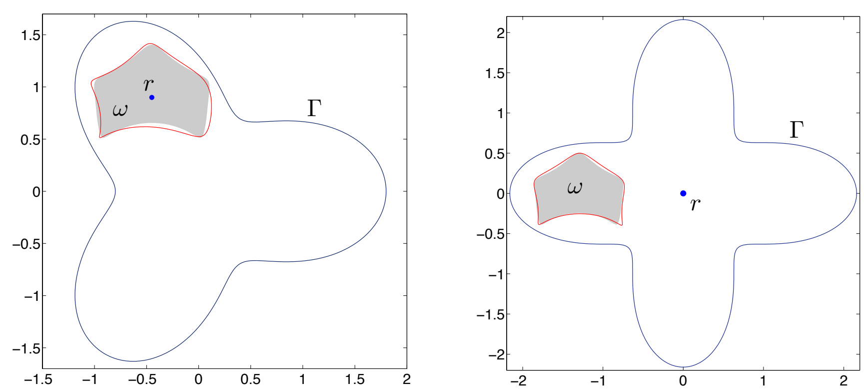

Figure 9: Other examples of reconstruction with more complex boundary Γ.

where e^f and v are both harmonic in $\Omega \setminus \overline{\omega}$ with Dirichlet data $e^f|_\Gamma = f$, $e^f|_\gamma = 0$ and $v|_\Gamma = 0$, $v|_\gamma = 1$. For every $w \in H^1_0(\Omega \setminus \overline{\omega})$, we get:

$$\int_{\Omega\setminus\overline{\omega}} |\nabla(w + e^f + cv)|^2 \mathrm{d}x = \int_{\Omega\setminus\overline{\omega}} |\nabla w|^2 \mathrm{d}x + \int_{\Omega\setminus\overline{\omega}} |\nabla e^f|^2 \mathrm{d}x + c^2 \int_{\Omega\setminus\overline{\omega}} |\nabla v|^2 \mathrm{d}x + 2c \int_\gamma \partial_n e^f \mathrm{d}\sigma,$$

and therefore the minimum in (A.3) is unique and achieved for $w = 0$ and

$$c^f = -\int_\gamma \partial_n e^f \mathrm{d}\sigma \left(\int_{\Omega\setminus\overline{\omega}} |\nabla v|^2 \mathrm{d}x \right)^{-1}.$$

The corresponding fonction $u^f := e^f + c^f v$ is the unique minimizer of problem (A.1), and can easily be shown to solve System (1.2). It is classical to verify that, reciprocally, every solution of System (1.2) provides a solution to the minimization problem (A.1).

Seeking the minimum of problem (A.2) in the form $u = u^f + w$ with $w \in H_0^1(\Omega)$, we are led to consider the new, equivalent, minimization problem:

$$\min_{w \in H_0^1(\Omega)} \int_{\Omega\setminus\overline{\omega}} |\nabla w|^2 \mathrm{d}x + \frac{\alpha}{2} \int_{\omega} |\nabla w|^2 \mathrm{d}x + \int_{\gamma} \partial_n u^f w \, \mathrm{d}\sigma, \tag{A.4}$$

where we have used the fact that $u^f = c^f$ in ω. The existence and uniqueness for such a problem is straightforward and we denote by w_α^f the minimizer. Introducing

$$\overline{w}_\alpha^f = \frac{1}{\mathrm{mes}(\omega)} \int_\omega w_\alpha^f \, \mathrm{d}x,$$

and taking into account the condition (1.2d), the last term in the right hand side can be rewritten as:

$$\int_\gamma \partial_n u^f w_\alpha^f \, \mathrm{d}\sigma = \int_\gamma \partial_n u^f (w_\alpha^f - \overline{w}_\alpha^f) \, \mathrm{d}\sigma.$$

Invoking the Poincaré-Wirtinger inequality in ω, we get the estimate:

$$\left| \int_\gamma \partial_n u^f w_\alpha^f \, \mathrm{d}\sigma \right| \leqslant C \|\nabla w_\alpha^f\|_{L^2(\omega)}, \tag{A.5}$$

where the constant $C > 0$ depends only upon ω. The minimum (A.4) is negative ($w = 0$ is an admissible function), whence we deduce that:

$$\frac{\alpha}{2} \|\nabla w_\alpha^f\|_{L^2(\omega)} \leqslant C,$$

and therefore

$$\|\nabla w_\alpha^f\|_{L^2(\omega)} \to 0 \quad \text{as} \quad \alpha \to +\infty. \tag{A.6}$$

Remarking again that the minimum (A.4) is negative and using the estimate (A.5) together with the convergence result (A.6), we deduce that:

$$\|\nabla w_\alpha^f\|_{L^2(\Omega\setminus\overline{\omega})} \to 0 \quad \text{as} \quad \alpha \to +\infty,$$

and the proof is completed. $\square$

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